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Deformations in topological string theory

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Topological Membranes and Boundary Strings

Based on the Deligne theorem one expects that the deformation theory of a two-dimensional topological field theories, or more generally of a closed string (field) theory, can be formulated in terms of a three-dimensional theory. Open membranes are then the three-dimensional objects to study.

A parallel question is what the effect is of the 3-form field on the closed string theory. Indeed the natural generalisation of the 2-form coupling to the bulk of the string is the 3-form field in the case of the open membrane. This 3-form field can be interpreted either as the field strength of a 2-form gauge field – which couples to the boundary string as a gauge field – or as the C-field in M-theory, or as the 3-form RR field in type IIA string theory. Attempts to describe the effect in terms of constrained canonical quantisation has been undertaken recently [52, 53, 54, 55]. In these papers a noncommutative deformation of loop space was suggested. A natural situation where the effect of a 3-form occurs is the M-theory membrane ending on a M5-brane. This situation is particularly relevant as it may provide more insight about the still mysterious M5-brane. The place to study the effects are various decoupling limits of the M5-brane theories, in particular the $(2, 0)$ little string theory [56, 57], and the recently proposed OM theory point [53, 58]. In these situations the decoupled theories one studies can be interpreted as closed string theories. Moreover, they can be seen as the boundary of the supermembrane. The C-field is a bulk membrane deformation. The effect of this C-field can therefore be interpreted as a deformation of a closed string by an open membrane. Related to this by double dimensional reduction is the Type IIA situation of a D2-brane ending on a D4-brane, in a certain decoupling limit.

In Chapter 3, we studied deformations of boundary theories for open strings by bulk operators. We found that the deformation theory of this 1-algebra indeed had the structure of a 2-algebra. This would lead us to expect that the 3-algebra deformation of the 2-algebra formed by the closed strings can be found in the context of open membranes. In this chapter we will argue that this is indeed the case.

We discuss topological open membranes in a general setting. We try to describe the deformation theory of the boundary string theory by the membrane bulk operators. Though we are not able to prove all Ward identities in detail, due to our lack of understanding of three-dimensional conformal field theories, we argue that indeed the L_∞ structure is deformed, and that the deformation theory has the structure of the second class of deformation complexes.

In Section 5.3 we describe an explicit example for the topological open membrane (TOM), which was defined in [59]: an open membrane with only a WZ term, defined by a closed 3-form field. The undeformed boundary string theory is the closed string version of the Cattaneo-Felder model. The relevant algebra will be the algebra of boundary operators, and it has the structure of a 2-algebra. The deforming algebra is the bulk algebra of the membrane. The coupling of the bulk membrane to the C-field indeed deforms the closed string Lie bracket. We find that it induces a trilinear operation, which gives a correction to the Jacobi identity of the bracket.

In Section 5.9 we mention some possible extensions and relations to physical models, such as OM theory, self-dual little strings, and M5-branes. On the basis of the structure that we found in the open membrane, we speculate about consistent generalisations of interacting 2-form gauge theories, such as 'non-abelian' 2-forms.

5.1. Three-Dimensional Topological Field Theories

Three-dimensional topological field theories can be treated in a manner quite similar to two-dimensional ones, so we will be quite brief here. There are three-point functions C_{ijk} defining a symmetric product, which are equivalent to the two-dimensional ones. Using a unit operator we define a metric by the two-point function equivalent to C_{0ij}. The bracket is now defined by the three-point functions

$$B_{ijk} = \Big\langle \phi_i \oint \phi_j^{(2)} \phi_k \Big\rangle, \tag{5.1}$$

where we integrate over a 2-sphere enclosing ϕ_k.

As for any TFT, we demand the presence of a BRST operator Q and of an operator G, such that $\{Q, G\} = d$. In the presence of a boundary, these operators also induce an action on the boundary operators, though in general there may be extra boundary terms. The symmetry current G in the topological open membrane induces a Ward identity of the form

$$\begin{aligned} 0 = &\sum_m \xi^\mu(x_m)\Big\langle \prod_n \phi_{i_n}(z_n)\alpha_{a_1}(x_1)\cdots G_\mu\alpha_{a_m}(x_m)\cdots\alpha_{a_r}(x_r)\Big\rangle \\ &+\sum_n \xi^\mu(z_n)\Big\langle \phi_{i_1}(z_1)\cdots G_\mu\phi_{i_n}(z_n)\cdots\phi_{i_s}(z_s)\prod_m \alpha_{a_m}(x_m)\Big\rangle, \end{aligned}$$

where the z's are points in the bulk and the x's are points on the boundary. Here the operators ϕ and α can be any operator, not necessarily BRST-closed. They can also be descendants. In this equation, ξ^μ is a globally defined conformal vector field. The conformal group of the 3-ball is $\mathrm{SO}(2,2)$, which is six-dimensional. Therefore, we have a basis of six vector fields to choose for the ξ's. This counting relies very much on a conformally invariant gauge-fixing of the open membrane. A priori we do not know if such a gauge-fixing does exist. In the following we will assume this.

We study deformations of the closed string correlation functions by including new operators ϕ_i in the correlation functions, which we view as deforming operators. However, these operators will now be bulk operators for the membrane. We can define mixed two-point functions by

$$\Phi_{ia} = \Big\langle \phi_i\alpha_a\Big\rangle. \tag{5.2}$$

The mixed three-point functions are defined by

$$\Phi_{iab} = \Big\langle \phi_i\alpha_a\int\alpha_b^{(2)}\Big\rangle. \tag{5.3}$$

We cannot have any correlators 'in between'; if we would insert a first descendant, integrated over a cycle, we could always shrink the cycle to zero. Higher mixed correlators are given by

$$\Phi_{ia_0a_1\dots a_n} = \Big\langle \phi_i\alpha_{a_0}\int\alpha_{a_1}^{(2)}\cdots\int\alpha_{a_n}^{(2)}\Big\rangle. \tag{5.4}$$

We will assume that the closed string Ward identities for G are still valid, so that these correlators are symmetric in the closed string indices. For the relevant situations, we will argue below that this is indeed the case.

When we introduce extra membrane operators in the Φ's, we should integrate them,

$$\Phi_{ija_0a_1\dots a_n} = \Big\langle \phi_i\int\phi_j^{(3)}\alpha_{a_0}\int\alpha_{a_1}^{(2)}\cdots\int\alpha_{a_n}^{(2)}\Big\rangle. \tag{5.5}$$

Now the algebra of deforming operators is assumed to have the same structure as the closed string theory. That is, we have Q and G. Also, these operators should be related to the corresponding operators on the closed string theory. The BRST operator of the open membrane descends to this boundary string, by integrating the corresponding current over a half-sphere enclosing the boundary operator. The correlators will also be symmetric in the i, j indices. This should also be true if we introduce extra integrated deforming operators. Indeed, the G operator is zero on these top forms. These assumptions imply that the mixed correlators are integrable: there are functions $\Phi_{a_0\ldots a_n}(t)$ such that $\Phi_{ia_0\ldots a_n}(t) = \partial_i \Phi_{a_0\ldots a_n}(t)$, where $\partial_i = \frac{\partial}{\partial t^i}$. The coefficients in the expansion in t are the higher correlation functions. We can therefore formally write these deformed correlators as

$$\Phi_{a_0 a_1 \ldots a_n}(t) = \Big\langle \alpha_{a_0} \int \alpha^{(1)}_{a_1} \alpha_{a_2} \int \alpha^{(2)}_{a_3} \cdots \int \alpha^{(2)}_{a_n} \, \mathrm{e}^{t^i \int \phi^{(3)}_i} \Big\rangle. \tag{5.6}$$

5.2. The Algebraic Structure of Open Membranes

The essential identity needed to view the insertions of bulk operators as a deformation of the boundary algebra was the symmetry of the higher correlators $\Phi_{ija_0a_1\ldots a_n}$ defined in (5.5), with respect to the bulk indices:

$$\Big\langle \phi_i \phi^{(3)}_j \alpha_a \Big\rangle = C \Big\langle \phi^{(3)}_i \phi_j \alpha_a \Big\rangle. \tag{5.7}$$

where C should be a function of the insertion points. In order for the integrated correlation functions to be truly invariant under this switch, this function should be the Jacobian of the coordinate transformation from the insertion point of ϕ_i to the insertion point of ϕ_j.[1] We will now argue that the assumption of conformal invariance gives enough global Ward identities to give the above relation at least. We are however not in a position to determine the factor C, due to a lack of understanding of the conformal invariance. Therefore the invariance of the integrated correlation functions will not be established completely. As we argued, assuming conformal invariance we have six independent Ward identities of the form (5.2). However, in the present case we do not want the boundary operator α_a to get involved. This can be established if the vector field ξ^μ used in the Ward identity vanishes at the insertion point of this operator. This gives two restrictions on ξ, leaving us with four Ward identities. These are sufficient to transfer the three independent components of G_μ from ϕ_j to ϕ_i, thereby establishing the existence of the above relation. As G vanishes on any second descendant of a boundary operator or a third descendant of a bulk operator, the relation remains true if we insert any number of these maximal descendants.

More important is a relation of the form

$$\Big\langle \int \phi^{(3)}_i \alpha_a \oint \alpha^{(1)}_b \alpha_c \Big\rangle = \Big\langle \phi_i \int \alpha^{(2)}_a \int \alpha^{(2)}_b \alpha_c \Big\rangle, \tag{5.8}$$

[1] Conformal invariance guarantees the existence of this coordinate transformation.

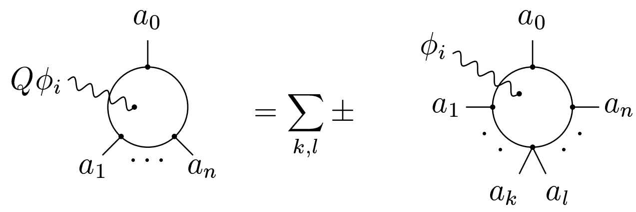

FIGURE 5.1: *Factorisation of the BRST operator.*

showing that we can interpret the mixed correlation functions as bulk to boundary metrics deformed by the boundary operators. This can be proved using Ward identities of the form

$$\left\langle \phi_i^{(3)} \alpha_a \alpha_b^{(1)} \alpha_c \right\rangle = C \left\langle \phi_i \alpha_a^{(2)} \alpha_b^{(2)} \alpha_c \right\rangle. \tag{5.9}$$

We start from the right-hand side. A priori, we have 6 independent global vector fields. Next we choose the vector fields that fix the position of α_c. As this gives two conditions, there are four vector fields. Of these, we use two vector fields to transfer the second descendant from α_b to ϕ_i. Next we choose the third vector field such that it fixes the position of α_b as well (as this gives two more conditions, there are two independent choices). We can use this to transfer one descendant of α_a to ϕ_i without getting additional terms. This argument shows that conformal invariance of the TOM theory is large enough to get this Ward identity (we only need 5 independent vector fields). Again, we cannot decide whether C is a Jacobian. We expect this to be true on the general basis of conformal invariance and will assume it henceforth. Equation (5.8) means that the correlator Φ_{iabc} is a deformation of the bracket. It would remain valid when we include extra fully integrated bulk and boundary insertions.

We want to view the mixed correlators as intertwiners between the closed membrane algebra and the deformations of the on-shell L_∞ structure, given by the boundary correlators $G_{abc\ldots}$. An essential structure of the topological bulk theory is the BRST operator. A BRST operator acting on the closed string operator in the mixed correlators can be deformed to a contour around the boundary operators. Using the descent equations for the boundary operators gives the following identity, also depicted in Figure 5.1.

$$\begin{aligned}
\left\langle Q\phi_i \alpha_{a_0} \int \alpha_{a_1}^{(2)} \cdots \int \alpha_{a_n}^{(2)} \right\rangle &= \left\langle \phi_i \{\alpha_{a_0}, \alpha_{a_1}\} \int \alpha_{a_2}^{(2)} \cdots \int \alpha_{a_n}^{(2)} \right\rangle \\
&\quad + (-1)^{n+1} \left\langle \phi_i \int \alpha_{a_1}^{(2)} \cdots \int \alpha_{a_{n-1}}^{(2)} \{\alpha_{a_n}, \alpha_{a_0}\} \right\rangle \\
&\quad + \sum_{k=1}^{n-1} (-1)^k \left\langle \phi_i \alpha_{a_0} \int \alpha_{a_1}^{(2)} \cdots \int \{\alpha_{a_k}, \alpha_{a_{k+1}}\}^{(2)} \cdots \int \alpha_{a_n}^{(2)} \right\rangle.
\end{aligned} \tag{5.10}$$

In this derivation, the boundary operators are taken on-shell (BRST-closed), while for

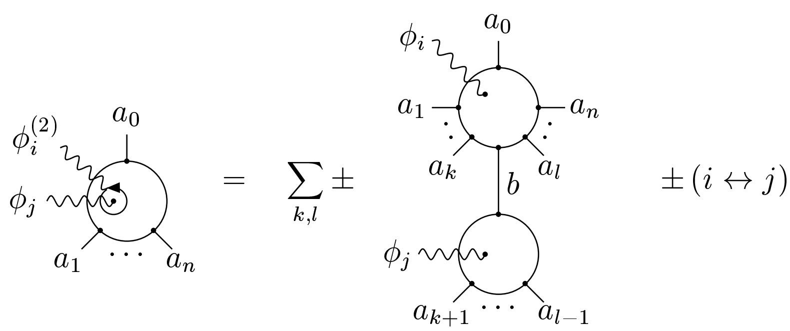

FIGURE 5.2: *Factorisation of the bracket.*

ϕ_i we take an arbitrary local membrane operator. The boundary terms in the factorised diagrams are related to points in the moduli space where two boundary operators approach each other. They arise from a total derivatives of the form $\int d\alpha_a^{(1)} = \int Q\alpha_a^{(2)}$. Its boundary term near another boundary operator will still contain a first descendant, which is integrated around the insertion point. Thus it involves the bracket rather than the product. We find that the bulk BRST operator corresponds to the operator δ_b. More generally, if we include off-shell boundary operators we find that there are corrections from the boundary BRST operator, which are easily seen to correspond to the coboundary δ_Q acting on the maps. The coboundary operator on the deformation complex is therefore found to be $\delta_Q + \delta_b$, which is indeed the coboundary operator related to deformations of the DL (or more generally of the L_∞) structure. Note that we do not consider deformations of the A_∞ structure. The mathematical reason is that δ_m has degree 0 here, so it has an interpretation of a 'gauge symmetry'. It seems to be more related to deformations of the open string; for the closed string it does not correspond to a physical operator. We can do the same with the inclusion of a second bulk operator, that is we look at the factorisation of the correlation function

$$\Big\langle \int (Q\phi_i)^{(3)} \int \phi_j^{(3)} \alpha_{a_0} \oint \alpha_{a_1}^{(1)} \alpha_{a_2} \int \alpha_{a_3}^{(2)} \cdots \int \alpha_{a_n}^{(2)} \Big\rangle, \tag{5.11}$$

which vanishes on-shell. The basic difference is that the undeformed products m (of any order) are replaced by the deformed products. Furthermore, there is an extra boundary term related to the two bulk operators coming close together. This involves the integral of the second descendant of ϕ_i around ϕ_j, because of the total derivative term $d\phi_i^{(2)}$ coming from pulling Q through the descendants. It gives the bracket in the membrane theory. The vanishing of (5.11) gives the relation depicted in Figure 5.2,

$$\Phi(\{\phi_i, \phi_j\}) = [\Phi(\phi_i), \Phi(\phi_j)]. \tag{5.12}$$

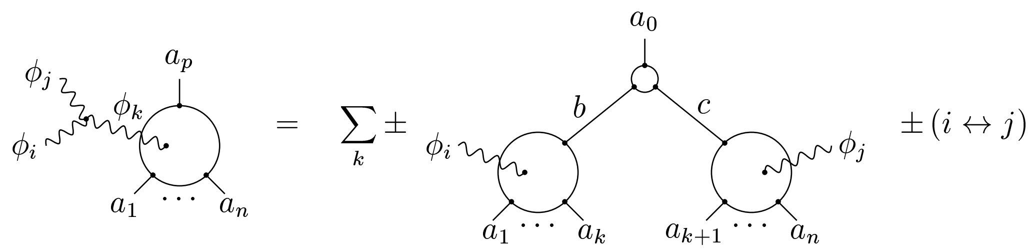

FIGURE 5.3: *Factorisation of the product.*

There is also a factorisation giving the bulk product as a boundary term, and several factorised correlation functions as the other boundary terms. However, it involves a codimension 3 boundary, starting from the deformed correlator with two deforming operators. This can be seen from the fact that we need to replace $\int \phi_i^{(3)} \int \phi_j^{(3)}$ by a single descendant $\int (\phi_i \cdot \phi_j)^{(3)}$. The factorisation is depicted in Figure 5.3. From the fact that we have a codimension 3 boundary, it can be seen that the undeformed factor involves the bracket of the boundary theory. This gives the following identity:

$$\Phi(\phi_i \cdot \phi_j)(\alpha_{a_1}, \ldots, \alpha_{a_n}) = \sum_k \pm \{\Phi_i(\alpha_{a_1}, \ldots, \alpha_{a_k}), \Phi_j(\alpha_{a_{k+1}}, \ldots, \alpha_{a_n})\}. \tag{5.13}$$

BRST operator, bracket and product constitute the afore-described Poisson algebra (3-algebra) of deformations of the L_∞ structure of closed strings. We will now turn to a concrete realisation.

5.3. BV QUANTISATION OF THE TOPOLOGICAL OPEN MEMBRANE

In this section we discuss as an example an explicit topological open membrane theory; this discussion is independent of the considerations of the previous section. The model we will study is the membrane with only a WZ term, whose action is given by

$$S = \frac{1}{6} \int_M C_{ijk} dX^i \wedge dX^j \wedge dX^k + \frac{1}{2} \int_{\partial M} B_{ij} dX^i \wedge dX^j, \tag{5.14}$$

M denoting an arbitrary target space. This action appears for example as a suitable decoupling limit of the open supermembrane in M-theory [52]. This action as it stands is ill-defined, since it is cubic (no perturbation theory) and dependent of the gauge fixing. In order to do calculations and to quantise the action, we use a first-order formalism and BV quantisation techniques [59]. The BV-quantised membrane theory is inspired by the Cattaneo-Felder model for the topological open string with a B-field WZ term, as discussed in Section 2.7.

The BV quantisation of the TOM theory defined by the WZ term goes very much along the same lines. x^μ and θ^μ are the coordinates on the membrane world-volume. We will show that the undeformed TOM is equivalent to the topological closed string theory given by CF. From the philosophy above, in order to construct the TOM we have to introduce two more sets of superfields, which we denote $\boldsymbol{\psi}^i$ and $\boldsymbol{F}_i$, which serve as conjugate momenta for the two superfields $\boldsymbol{X}^i$ and $\boldsymbol{\chi}_i$. The four superfields describing the TOM can be expanded as

$$\begin{aligned}
\boldsymbol{X}^i &= X^i + \rho^i_\mu\theta^\mu + \frac{1}{2}X^i_{\mu\nu}\theta^\mu\theta^\nu + \frac{1}{6}\rho^i_{\mu\nu\lambda}\theta^\mu\theta^\nu\theta^\lambda, \\
\boldsymbol{\chi}_i &= \chi_i + H_{i\mu}\theta^\mu + \frac{1}{2}\chi_{i\mu\nu}\theta^\mu\theta^\nu + \frac{1}{6}H_{i\mu\nu\lambda}\theta^\mu\theta^\nu\theta^\lambda, \\
\boldsymbol{\psi}^i &= \psi^i + A^i_\mu\theta^\mu + \frac{1}{2}\psi^i_{\mu\nu}\theta^\mu\theta^\nu + \frac{1}{6}A^i_{\mu\nu\lambda}\theta^\mu\theta^\nu\theta^\lambda, \\
\boldsymbol{F}_i &= F_i + \eta_{i\mu}\theta^\mu + \frac{1}{2}F_{i\mu\nu}\theta^\mu\theta^\nu + \frac{1}{6}\eta_{i\mu\nu\lambda}\theta^\mu\theta^\nu\theta^\lambda.
\end{aligned} \tag{5.15}$$

These fields have ghost degrees 0, 1, 1, and 2, respectively. This will give a nontrivial bracket, of degree -1. Here Π is an operator that shifts the degree in the fibre by one, see Section 2.4; it gives $\boldsymbol{\chi}$ degree 1. The scalar components $(X^i, \chi_i, \psi^i, F_i)$ can be viewed as coordinates on the superspace $\Pi T(\Pi T^*M)$, when $\boldsymbol{\psi}, \boldsymbol{F}$ are the conjugates of $\boldsymbol{X}, \boldsymbol{\chi}$ respectively. For this space one can equivalently read $T^*[2](\Pi T^*M)$, when one considers $\boldsymbol{F}, \boldsymbol{\psi}$ to be the conjugates of $\boldsymbol{X}, \boldsymbol{\chi}$ respectively $(0+2=2,\ -1+2=1)$. The former is convenient for generalisation, as one can apply ΠT iteratively to generalise to higher dimensions [59], while the latter is more insightful for our present purposes, since it will give us a symplectic structure of degree 2.

Viewing (x^μ, θ^μ) as coordinates on the superspace ΠTN, where N is the world-volume of the membrane, the four fields can be viewed as parametrising a map $\Pi TN \to \Pi T(\Pi T^*M)$ between the two superspaces. We will choose boundary conditions such that the new fields $\boldsymbol{\psi}^i$ and $\boldsymbol{F}_i$, mapping to the fibre, vanish on ∂N (Dirichlet boundary conditions). This means that the boundary ∂N maps to the base space ΠT^*M of the target space, parametrised by X^i and χ_i.

In order to get a BV quantisation, we need to introduce a BV (anti)bracket. From our motivation of choosing $\boldsymbol{F}$ and $\boldsymbol{\psi}$ as the 'momenta' of the superfields $\boldsymbol{X}$ and $\boldsymbol{\chi}$, we have a natural odd symplectic structure on the superfields above,

$$\boldsymbol{\omega}_{BV} = \int_N \int d^3\theta \Big(\delta\boldsymbol{X}^i\delta\boldsymbol{F}_i + \delta\boldsymbol{\chi}_i\delta\boldsymbol{\psi}^i\Big), \tag{5.16}$$

where δ denotes the d-operator (de Rham differential) on field space. This is a symplectic form of ghost degree -1. This symplectic structure defines the BV bracket, which is dual to it, and can formally be written

$$(\cdot,\cdot)_{BV} = \frac{\partial}{\partial \boldsymbol{X}^i} \wedge \frac{\partial}{\partial \boldsymbol{F}_i} + \frac{\partial}{\partial \boldsymbol{\chi}_i} \wedge \frac{\partial}{\partial \boldsymbol{\psi}^i}. \tag{5.17}$$

This is seen to derive from the BV operator $\triangle = \frac{\partial^2}{\partial \boldsymbol{X}^i \partial \boldsymbol{F}_i} + \frac{\partial^2}{\partial \boldsymbol{\chi}_i \partial \boldsymbol{\psi}^i}$.

The BV action of the undeformed theory is given by the terms of total ghost degree 3,

$$S_0 = \int_N \int d^3\theta \Big(\boldsymbol{F}_i D\boldsymbol{X}^i + \boldsymbol{\psi}^i D\boldsymbol{\chi}_i + \boldsymbol{F}_i \boldsymbol{\psi}^i \Big), \tag{5.18}$$

where $D = \theta^\mu \partial_\mu$ is a representation of the d-operator on target space. It is easily seen that the BV action above satisfies both the classical and the quantum master equation, $\triangle S_0 = (S_0, S_0)_{BV} = 0$. It determines the BRST operator through $\boldsymbol{Q}_0 = (S_0, \cdot)_{BV}$,

$$\boldsymbol{Q}_0 = (D\boldsymbol{X}^i + \boldsymbol{\psi}^i)\frac{\partial}{\partial \boldsymbol{X}^i} + D\boldsymbol{\psi}^i \frac{\partial}{\partial \boldsymbol{\psi}^i} + (D\boldsymbol{\chi}_i + \boldsymbol{F}_i)\frac{\partial}{\partial \boldsymbol{\chi}_i} + D\boldsymbol{F}_i \frac{\partial}{\partial \boldsymbol{F}_i}. \tag{5.19}$$

It is now possible to construct descendants of operators. For this we use the operator $\boldsymbol{G}_\mu$, which acts as $\frac{\partial}{\partial \theta^\mu}$ on superfields. It satisfies the anticommutation relation $\{\boldsymbol{Q}_0, \boldsymbol{G}_\mu\} = \partial_\mu$. In general, an operator is constructed by starting with a function on $T^*[2](\Pi T^* M)$, $f(X, \chi, \psi, F)$. The full observable is the pull-back to ΠTN using the superfields in $\boldsymbol{\phi} = (\boldsymbol{X}, \boldsymbol{\chi}, \boldsymbol{\psi}, \boldsymbol{F})$, which we denote $\boldsymbol{\mathcal{O}}_f(\boldsymbol{X}, \boldsymbol{\chi}, \boldsymbol{\psi}, \boldsymbol{F}) = f(\boldsymbol{X}, \boldsymbol{\chi}, \boldsymbol{\psi}, \boldsymbol{F}) = \boldsymbol{f}$. The components found by expanding in powers of θ are the descendants of the scalar part. In general, the BRST operator will have the form

$$\boldsymbol{Q} = D + \boldsymbol{\phi}^* Q, \tag{5.20}$$

where Q is an odd algebraic operator on the target space. The terms come from the kinetic part and the rest, which does not contain any derivatives. This automatically satisfies the correct anticommutation relation with $\boldsymbol{G}$.

Motivated by CF, we will now turn on a B-field on the boundary. This is done by shifting some of the fields

$$\boldsymbol{F}_i \to \boldsymbol{F}_i - \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{X}^i}, \qquad \boldsymbol{\psi}^i \to \boldsymbol{\psi}^i - \frac{\partial \boldsymbol{\beta}}{\partial \boldsymbol{\chi}_i}, \tag{5.21}$$

where $\boldsymbol{b}$ is a ghost number two operator of the form

$$\boldsymbol{\beta} = \frac{1}{2} \boldsymbol{b}^{ij}(\boldsymbol{X}) \boldsymbol{\chi}_i \boldsymbol{\chi}_j. \tag{5.22}$$

Here $\boldsymbol{b}^{ij}(X)$ can be any bivector on the space M, or we can consider its pull-back by the superfield $\boldsymbol{X}$ to ΠTN, $\boldsymbol{b}^{ij}(x, \theta) = b^{ij}(\boldsymbol{X}(x, \theta))$. This can be seen as a gauge transformation in the fibre of the space $T^*[2](\Pi T^* M)$. In general, $\boldsymbol{b}^{ij}$ has a field strength given by

$$\boldsymbol{h}^{ijk} = \boldsymbol{b}^{il}\partial_l \boldsymbol{b}^{jk} + \boldsymbol{b}^{jl}\partial_l \boldsymbol{b}^{ki} + \boldsymbol{b}^{kl}\partial_l \boldsymbol{b}^{ij}. \tag{5.23}$$

For a Poisson bivector b^{ij} on M this field strength vanishes. This corresponds to the B-field in noncommutative open string theory being constant.

Performing the above shifts, the bulk part of the undeformed BV action in the presence of a $\boldsymbol{b}$-field becomes

$$S = \int_N \int d^3\theta \Big(\boldsymbol{F}_i D\boldsymbol{X}^i + \boldsymbol{\psi}^i D\boldsymbol{\chi}_i + \boldsymbol{F}_i \boldsymbol{\psi}^i - \boldsymbol{b}^{ij}\boldsymbol{F}_i\boldsymbol{\chi}_j - \frac{1}{2}\partial_k \boldsymbol{b}^{ij}\boldsymbol{\psi}^k\boldsymbol{\chi}_i\boldsymbol{\chi}_j - \frac{1}{6}\boldsymbol{h}^{ijk}\boldsymbol{\chi}_i\boldsymbol{\chi}_j\boldsymbol{\chi}_k \Big). \tag{5.24}$$

We leave out the boundary term, because we are only interested in the bulk action. It satisfies both the classical and the quantum master equation precisely if $\boldsymbol{h} = 0$. The b-term gives an infinite constant in the quantum part master; in an approriate regularisation scheme however, the prefactor vanishes before this constant tends to infinity [7]. Because the auxiliary fields $\boldsymbol{F}_i$ appear only linearly, we can exactly integrate them out, which gives a delta function. After solving for $\boldsymbol{\psi}_i$ in this equation, the action reduces to a pure boundary term in the case of a Poisson bivector $\boldsymbol{b}^{ij}$,

$$S_{CF} = \int_{\partial N} \int d^2\theta \Big(\boldsymbol{\chi}_i D\boldsymbol{X}^i + \frac{1}{2}\boldsymbol{b}^{ij}\boldsymbol{\chi}_i\boldsymbol{\chi}_j \Big) = \int_{\partial N} \Big(H_i dX^i + \chi_i d\rho^i + \frac{1}{2} b^{ij} H_i H_j + \cdots \Big). \tag{5.25}$$

This is precisely the action of the CF model, as announced. This is related to the usual topological closed string with just the B-field WZ term by integrating out H. H is the equivalent of η in (2.64) in a local trivialisation. Note that H is a mapping between the fibres, while in the CF model η the base point is taken into consideration. This is why η is a section of $X^*(T^*M) \otimes T^*N$.

The boundary operators are determined by functions f of the scalar fields X^i and χ_i, that is functions on the base space ΠT^*M of the target space. The corresponding boundary operator α_f and its descendants combined as

$$\alpha_f + \theta\alpha_f^{(1)} + \frac{1}{2}\theta^2\alpha_f^{(2)} = f(\boldsymbol{X}, \boldsymbol{\chi}). \tag{5.26}$$

We will sometimes denote this by $\boldsymbol{f}$. It is natural to view the space of functions on ΠT^*M as the polynomial algebra $\mathbb{C}[\{X^i\},\{\chi_i\}]$ generated by X^i and χ_i. By formally replacing the fermionic generators χ_i by the basic vector fields ∂_i, one sees that the boundary operators are in one-to-one correspondence with the multivector fields on the target space M. Our undeformed boundary algebra A will thus be the algebra of multivector fields $A = \Gamma(M, \bigwedge^* TM)$.

The three-point functions determine a structure of an algebra on these boundary operators, which indeed turns out to be a 2-algebra. More precisely, for the product and the bracket this relation will be given by

$$\Big\langle \alpha_\delta \alpha_f \alpha_g \Big\rangle \equiv \Big\langle \alpha_\delta \alpha_{f\cdot g} \Big\rangle, \qquad \Big\langle \alpha_\delta \oint \alpha_f^{(1)} \alpha_g \Big\rangle \equiv \Big\langle \alpha_\delta \alpha_{\{f,g\}} \Big\rangle, \tag{5.27}$$

where we took the outgoing state corresponding to a δ-function on the target space. The product is easily seen to be the wedge product on the multivector fields.

We will now calculate the bracket. A boundary operator is given by a function of X and χ, $f = f(X, \chi)$. This is the zeroth descendant. The descent equation gives for the first descendant

$$f^{(1)} = \rho^i \frac{\partial f}{\partial X^i} + H_i \frac{\partial f}{\partial \chi_i}. \tag{5.28}$$

The Gerstenhaber bracket of the boundary string is given by

$$\{f, g\} = \oint f^{(1)} g = \oint \Big(\rho^i \frac{\partial f}{\partial X^i}(X, \chi) + H_i \frac{\partial f}{\partial \chi_i}(X, \chi) \Big) g(X, \chi) \tag{5.29}$$

From this we can calculate the first order-corrected bracket, by calculating the corresponding correlation functions. One way is to use the boundary action (5.25) and to quantise canonically. One wants to identify the bracket with the quantum commutator; this is given in first order by the Poisson bracket. For canonical quantisation, we need to choose a time coordinate. Furthermore, we write $H_i = H_{1i}$, and $\rho^i = \rho^i_1$. The Poisson brackets are computed from the action,

$$\{H_i(u), X^j(v)\}_P = \delta_i^j \delta(u - v), \tag{5.30}$$
$$\{\rho^i(u), \chi_j(v)\}_P = \delta_j^i \delta(u - v). \tag{5.31}$$

By splitting the contour in (5.29) into two pieces according to the time, we find that we can write it in terms of a commutator. Analogously to (4.21) we find the bracket,

$$\{f, g\} = \frac{\partial f}{\partial X^i} \frac{\partial g}{\partial \chi_i} + (-1)^{|f|} \frac{\partial f}{\partial \chi_i} \frac{\partial g}{\partial X^i}. \tag{5.32}$$

This is thus a consequence of the $\boldsymbol{\psi}^i \boldsymbol{F}_i$ term in the BV action. It is precisely the Schouten-Nijenhuis bracket.

When b is nonzero, there is also a differential, so that the boundary string theory is a differential Gerstenhaber algebra. This differential is given by the BRST operator restricted to the boundary,

$$Qf = b^{ij} \chi_j \frac{\partial f}{\partial X^i} + \frac{1}{2} \partial_i b^{jk} \chi_j \chi_k \frac{\partial f}{\partial \chi_i}. \tag{5.33}$$

It is easily checked that Q is nilpotent if b^{ij} is a Poisson structure.

We conclude that the undeformed topological open membrane we proposed above reduces to a boundary theory when the master equation holds, and that it is given by the algebra of multivector fields. It is supplied with the differential Q above, the wedge product and the Schouten-Nijenhuis bracket, which indeed makes it into a 2-algebra. Our next task is to study the deformation of this 2-algebra. We first propose a natural deformation in the context of our BV-quantised theory.

5.4. Deformations of the TOM

The boundary string theory will be deformed by coupling the TOM to a bulk operator. We can construct bulk operators corresponding to functions $f(X, \chi, \psi, F)$ on $\Pi T(\Pi T^* M)$. They are given by the pull-back to the world-volume $\boldsymbol{f} = f(\boldsymbol{X}, \boldsymbol{\chi}, \boldsymbol{\psi}, \boldsymbol{F})$, using the superfields. This generates all descendants of the operator and to conserve ghost number, this should have degree 3. The natural topological deformation is to turn on a 3-form deformation in the bulk of the open membrane theory. Given a 3-form c this defines an operator ϕ_c, which for $b = 0$ is given by

$$\int_N \phi_c^{(3)} = \frac{1}{6} \int_N \int d^3\theta \, \boldsymbol{c}_{ijk} \boldsymbol{\psi}^i \boldsymbol{\psi}^j \boldsymbol{\psi}^k. \tag{5.34}$$

We will use this operator as the deformation of the BV action functional. It will turn out that the b-field in general does not have to define a strict Poisson structure in the deformed case, so we will not impose this as a requirement. The totally deformed BV action, including b, becomes

$$\begin{aligned} S = \int_N \int d^3\theta \Big(& \boldsymbol{F}_i D\boldsymbol{X}^i + \boldsymbol{\psi}^i D\boldsymbol{\chi}_i + \boldsymbol{F}_i \boldsymbol{\psi}^i + \boldsymbol{b}^{ij} \boldsymbol{F}_i \boldsymbol{\chi}_j + \frac{1}{2} \partial_k \boldsymbol{b}^{ij} \boldsymbol{\psi}^k \boldsymbol{\chi}_i \boldsymbol{\chi}_j + \frac{1}{6} \boldsymbol{h}^{ijk} \boldsymbol{\chi}_i \boldsymbol{\chi}_j \boldsymbol{\chi}_k \\ & + \frac{1}{6} \boldsymbol{c}_{ijk} (\boldsymbol{\psi}^i + \boldsymbol{b}^{il} \boldsymbol{\chi}_l)(\boldsymbol{\psi}^j + \boldsymbol{b}^{jm} \boldsymbol{\chi}_m)(\boldsymbol{\psi}^k + \boldsymbol{b}^{kn} \boldsymbol{\chi}_n) \Big). \end{aligned} \tag{5.35}$$

This action still satisfies the BV master equation, both the classical and the quantum, if the deformed field strength given by

$$\boldsymbol{h}_{\boldsymbol{c}}^{ijk} = \boldsymbol{b}^{il} \partial_l \boldsymbol{b}^{jk} + \boldsymbol{b}^{jl} \partial_l \boldsymbol{b}^{ki} + \boldsymbol{b}^{kl} \partial_l \boldsymbol{b}^{ij} + \boldsymbol{b}^{il} \boldsymbol{b}^{jm} \boldsymbol{b}^{kn} \boldsymbol{c}_{lmn}, \tag{5.36}$$

vanishes; this condition is only relevant for the open membrane. In the bulk it is an exact canonical transformation, for which it does not matter what $\boldsymbol{b}$ is. A pure bulk term satisfies the BV equation if it satisfies the master equation for the closed membrane, and if it vanishes on the boundary. This is because of the form of $\boldsymbol{Q}$, (5.20). Notice that this implies that b^{ij} is not necessarily a Poisson structure. Also note that we cannot invert $\boldsymbol{b}$ to find globally $dB = -C$, since the latter relation would imply that B is not invertible. Moreover, a closed C would be trivial. If we integrate out $\boldsymbol{F}$ while (5.36) vanishes, the action (5.35) reduces to a boundary theory with the WZ term (5.14) of the c-field,

$$S = \int_\Sigma \int d^2\theta \Big(\boldsymbol{\chi}_i D\boldsymbol{X}^i + \frac{1}{2} \boldsymbol{b}^{ij} \boldsymbol{\chi}_i \boldsymbol{\chi}_j \Big) + \frac{1}{6} \int_M \int d^3\theta \, \boldsymbol{c}_{ijk} D\boldsymbol{X}^i D\boldsymbol{X}^j D\boldsymbol{X}^k, \tag{5.37}$$

or written out in components,

$$S = \int_\Sigma \Big(H_i dX^i + \chi_i d\rho^i + \frac{1}{2} b^{ij} H_i H_j + b^{ij} \chi_i \chi_{(2)j} \Big) + \frac{1}{6} \int_M c_{ijk} dX^i dX^j dX^k. \tag{5.38}$$

This motivates our choice for the deforming operator, and for the whole model, since it shows that the model serves as a well-defined quantum action for the ill-defined theory based on the WZ term.

To calculate the first-order corrections to the algebraic structure we need to calculate the corresponding correlation functions, which define the map Φ_c corresponding to the operator ϕ_c. This can be related to a deformation on the algebra of multivector fields. For example, we can write

$$\Phi_c(\alpha_f, \alpha_g) \equiv \alpha_{\{f,g\}_1}, \tag{5.39}$$

where $\{\cdot,\cdot\}_1$ is the first-order deformation of the bracket on the multivector fields. In the next subsection we will use the Hochschild complex to calculate the effect on the algebra, at least in a first-order quantisation. The field theory we now have can in principle be used to calculate the correspondence of the Hochschild cohomology – the deforming operators – and the Hochschild complex – the differential operators – as a perturbation series in c (formality).

5.5. Hochschild Cohomology of the Boundary Closed String

In Section 4.3, we saw that the possible deformers are essentially given by elements of the Hochschild cohomology. We will now calculate this cohomology for the topological open membrane theory. We start with the situation $b = 0$.

We saw that the operators of the boundary closed string form the algebra of functions on ΠT^*M, which we represent by the algebra of polynomials $A = \mathbb{C}[\{X^i\}, \{\chi_i\}]$. As explained above, this corresponds to the algebra of multivector fields $\Gamma(M, \bigwedge^* TM)$. This is naturally a graded algebra, with the degree corresponding to the vector degree. This means that the generators X^i have degree 0, and χ_i have degree 1. This algebra indeed has the structure of a Gerstenhaber algebra or 2-algebra, with the product m given by the wedge product and the bracket b given by the Schouten-Nijenhuis bracket, defined by

$$\{\alpha, \beta\} = \frac{\partial\alpha}{\partial X^i}\frac{\partial\beta}{\partial\chi_i} + (-1)^{|\alpha|}\frac{\partial\alpha}{\partial\chi_i}\frac{\partial\beta}{\partial X^i}. \tag{5.40}$$

This bracket is similar to (5.32). This is the simplest nontrivial 2-algebra one can construct.

The deformation of the Gerstenhaber algebra of multivector fields is determined by the Hochschild cohomology. The Hochschild complex is the algebra of multi-differential operators. On the cohomology act three differentials, two of them related to the product structures on A. Taking the partial cohomology with respect to the differential δ_m

associated to the ordinary product, we can describe these multi-differential operators by introducing anticommuting coordinates ψ^i, representing ∂_{χ_i}, and commuting variables F_i, representing ∂_{X^i}. The physical interpretation of taking the cohomology with respect to δ_m is not completely clear. Because δ_m has degree 0, it can be viewed as a kind of symmetry.

We can give an explicit description of the Hochschild cohomology of a general polynomial algebra. Consider the algebra of polynomials in a finite number of $\mathbb{Z}$-graded variables x^i of degree $\deg(x^i) = q_i \in \mathbb{Z}$, so the space $A = \mathbb{C}[x^1, \ldots, x^N]$. We view it as an algebra over the operad $H_*(C_d)$ (see [6]) so a d-algebra, with zero differential and zero Lie bracket. Here we assume that $d \geq 2$. The Hochschild cohomology of this algebra is, as a $\mathbb{Z}$-graded vector space, the algebra of polynomials

$$H^*(\mathsf{Hoch}(A)) = \mathbb{C}[x^1, \ldots, x^N, y_1, \ldots, y_N] \tag{5.41}$$

in the doubled set of variables x^i, y_i, where the extra generators have degree $\deg(y_i) = d - q_i$ [6]. In general, for the algebra $\mathcal{O}(M)$ of regular functions on a smooth $\mathbb{Z}$-graded algebraic supermanifold M, the Hochschild cohomology is given by the algebra of functions on the total space of the twisted by $[d]$ cotangent bundle to M,

$$H^*(\mathsf{Hoch}(\mathcal{O}(M))) = \mathcal{O}(T^*[d]M). \tag{5.42}$$

The proof goes along the same lines as the Hochschild-Kostant-Rosenberg theorem, which gives this result for associative algebras of functions ($d = 1$).

When the Lie bracket on the original d-algebra is nonzero, this leads to a coboundary operator on the above Hochschild cohomology. To find the actual Hochschild cohomology one should take the cohomology with respect to this coboundary. This coboundary operator is canonically related to the bracket. A bracket on the d-algebra corresponds to a Poisson structure ω^{ij} of degree $1 - d$ on M. When we use local coordinates (x^i, y_i) on $T^*[d]M$, as in the polynomial algebras above, the coboundary operator is given locally by $\omega^{ij} y_j \frac{\partial}{\partial x^i}$, which indeed has degree 1. We can also give this differential operator globally on $T^*[d]M$. We denote the pull-back of the Poisson structure ω to the full space also by ω. The total space $T^*[p]M$ has a canonical 1-form θ. This 1-form is such that the canonical symplectic structure is given by $d\theta$, and in local coordinates is given by $\theta = y_i dx^i$ (this differential form might be familiar from classical mechanics, where it is usually denoted $p_i dq^i$). Contracting the bi-vector ω with this form leads to a vector field $\theta \cdot \omega$, generating the above differential.

Specialising to our case, the Hochschild cohomology can be described as a polynomial algebra: $H^*_{\delta_m}(\mathsf{Hoch}(A)) = \mathbb{C}[\{X^i\}, \{\chi_i\}, \{\psi^i\}, \{F_i\}]$. The degree of the generators ψ^i is 1, while the degree of F_i should be taken 2. These correspond precisely to the extra fields in the BV action. This identifies the Hochschild cohomology of the boundary topological closed string with the BV BRST cohomology of the topological open string. We will make some more remarks about this in the next subsection.

There is still a differential left, related to the bracket. It is defined in a similar way to the Gerstenhaber differential, but with the product replaced by the bracket. This differential is easily calculated on the above polynomial algebra to be given by

$$\delta_b = \psi^i \frac{\partial}{\partial X^i} + F_i \frac{\partial}{\partial \chi_i}, \tag{5.43}$$

which correctly has degree 1. The full Hochschild cohomology is now the cohomology of the above polynomial algebra with respect to this differential. This algebra has a natural Poisson structure of degree -2, given by

$$\{\alpha, \beta\} = \frac{\partial \alpha}{\partial X^i}\frac{\partial \beta}{\partial F_i} - (-1)^{|\alpha|}\frac{\partial \alpha}{\partial \chi_i}\frac{\partial \beta}{\partial \psi^i} \pm (\alpha \leftrightarrow \beta). \tag{5.44}$$

This is to be compared with (5.17). The structure of the differential δ_b, the bracket of degree -2 and the product makes the Hochschild cohomology into a 3-algebra, which is just a differential Poisson algebra, except from the degree of the bracket.

The cohomology with respect to the differential δ_b removes all dependence on χ and F, so that in the end we are left with only polynomials of X^i and ψ^i. These are the right generators, because we want to reconstruct the De Rham cohomology. Hence the cohomology equals that of the differential forms on $\mathbb{R}^n$. Note that the Poisson bracket of the 3-algebra is identically zero on the cohomology.

In general, for $A = \Gamma(M, \bigwedge^* TM)$, we find $H^* \mathsf{Hoch}(A) = H^*(M)$. This means that for sufficiently large p , we have $H^p(\mathsf{Def}(A)) = H^{p+2}(M)$. Especially, $H^1(\mathsf{Def}(A)) = H^3(M)$. This term in the complex determines the actual deformations. The element in the Hochschild cohomology corresponding to a closed 3-form c is represented by the polynomial $\frac{1}{6}c_{ijk}(X)\psi^i\psi^j\psi^k$. We are of course interested in the corresponding element in the full Hochschild complex, that is the map Φ_c deforming the algebra. To find it, remember that ψ^i corresponds to the operator $\frac{\partial}{\partial \chi_i}$ in the complex. This corresponds to a naive canonical quantisation, which gives

$$\frac{1}{6}c_{ijk}(X)\frac{\partial}{\partial \chi_i} \wedge \frac{\partial}{\partial \chi_j} \wedge \frac{\partial}{\partial \chi_k}. \tag{5.45}$$

Of course, this is only the leading term in the map from the Hochschild cohomology to the complex.[2] Notice that this is a trilinear differential operator. This means that a trilinear product in the L_∞ algebra is deformed.

[2]It should be compared with the leading term $b^{ij}\partial_i \wedge \partial_j$ for the deformation of the product in non-commutative geometry.

A coboundary δ_Q is now introduced by turning on a b-field b^{ij}, which we will take constant for simplicity. Otherwise extra derivatives of b are introduced, but the argument is identical. This introduces a derivation Q on the algebra, and the calculation of the cohomology for the double complex is more complicated, as we now have two coboundary operators δ_Q and δ_b on the complex. The total coboundary operator on the double complex $C = \mathsf{Hoch}(A)$ is given by $D = d + \delta = \delta_b + \delta_Q$. With both differentials nonzero, we can in general calculate the cohomology using spectral sequence techniques, see Section 5.10. This basically amounts to solving a series of descent equations. Starting from a class d-closed element α_0, we have descent equations $\delta\alpha_0 = -d\alpha_1$, etcetera. The two coboundary operators on the double complex are given by

$$\delta \equiv \delta_Q = b^{ij}\chi_j \frac{\partial}{\partial X^i} + b^{ij} F_j \frac{\partial}{\partial \psi^i}, \qquad d \equiv \delta_b = \psi^i \frac{\partial}{\partial X^i} + F_i \frac{\partial}{\partial \chi_i}. \tag{5.46}$$

It turns out that the descent equations can be solved introducing the following operator

$$\gamma = b^{ij}\chi_j \frac{\partial}{\partial \psi^i}. \tag{5.47}$$

It is easily checked that $[d, \gamma] = -\delta$. Note that this is the analogue of $\{Q, G\} = T$. It can be used to solve $\alpha_1 = \gamma\alpha_0$, $\alpha_2 = \gamma\alpha_1$, and so on.

Let us see what this implies for the deformation term, when we turn b on. First note that the operator d is not affected by turning on b, therefore we still conclude that the d-cohomology class α_0 is represented by an element $\frac{1}{p!}\alpha_{i_1 \dots i_p}(X)\psi^{i_1}\cdots\psi^{i_p}$, where $\alpha_{i_1\dots i_p}(X)$ is a closed p-form. The effect of γ is to replace ψ^i by $b^{ij}\chi_j$. Therefore, the total class α is given in terms of the same form, but with ψ^i replaced by $\psi^i + b^{ij}\chi_j$,

$$\alpha = \frac{1}{p!}\alpha_{i_1\dots i_p}(X)(\psi^{i_1} + b^{i_1 j_1}\chi_{j_1})\cdots(\psi^{i_p} + b^{i_p j_p}\chi_{j_p}). \tag{5.48}$$

Most interestingly, the class in the third cohomology related to the closed 3-form c is given by

$$\frac{1}{6}c_{ijk}\psi^i\psi^j\psi^k + \frac{1}{2}c_{ijk}b^{il}\chi_l\psi^j\psi^k + \frac{1}{2}c_{ijk}b^{il}b^{jm}\chi_l\chi_m\psi^k + \frac{1}{6}c_{ijk}b^{il}b^{jm}b^{kn}\chi_l\chi_m\chi_n. \tag{5.49}$$

This corresponds precisely to the deformation term in the action (5.35).

Using the first-order map (the 'quantisation') from the cohomology to the Hochschild complex, this translates into the following set of deformed operations in the algebra:

$$\begin{aligned}
Q \equiv b_1 &= b^{ij}\chi_j \frac{\partial}{\partial X^i} + \frac{1}{2}(\partial_k b^{ij} + c_{klm}b^{li}b^{mj})\chi_i\chi_j \frac{\partial}{\partial \chi_k} + \mathcal{O}(c^2), \\
\{\cdot,\cdot\} \equiv b_2 &= \frac{\partial}{\partial X^i} \wedge \frac{\partial}{\partial \chi_i} + \frac{1}{2}c_{ijk}b^{kl}\chi_l \frac{\partial}{\partial \chi_i} \wedge \frac{\partial}{\partial \chi_j} + \mathcal{O}(c^2), \\
\{\cdot,\cdot,\cdot\} \equiv b_3 &= \frac{1}{6}c_{ijk}\frac{\partial}{\partial \chi_i} \wedge \frac{\partial}{\partial \chi_j} \wedge \frac{\partial}{\partial \chi_k} + \mathcal{O}(c^2).
\end{aligned} \tag{5.50}$$

This changes the Jacobi identity off-shell as

$$\{f,\{g,h\}\} \pm \{g,\{h,f\}\} \pm \{h,\{f,g\}\} = Q\{f,g,h\} + \{Qf,g,h\} \pm \{f,Qg,h\} \pm \{f,g,Qh\}. \tag{5.51}$$

The master equation is a consequence of the BV master equation. If we are on cohomology, the correction vanishes. The corrections to the BRST operator Q and the bracket $\{\cdot,\cdot\}$ are precisely given by the corrections in the higher terms of the spectral sequence: the sum is simply the quantisation of the total representative. These operations satisfy the relations of a 'L_3 algebra'. Together with the undeformed product, it satisfies the relation of a 'G_3 algebra', if one defines this appropriately.

We conclude with a remark on the boundary conditions. Letting $\boldsymbol{X}$ and $\boldsymbol{\chi}$ vanish on ∂N, is a particular choice. Boundary conditions are determined by a choice of a Lagrangian subspace $L = \Pi T^*M \subset T^*[2](\Pi T^*M)$. The boundary condition for the scalars is such that the boundary of the super-world-volume $\Pi T(\partial N)$ is mapped into L. For different choices of L one can arrive at the A-model, the B-model, or the Cattaneo-Felder model on the boundary [59]. It is also possible to derive a more general boundary master equation.

5.6. Sigma Model Computations

The operations in this section can be calculated more concretely by computations of the corresponding membrane correlators. Direct tree-level computations can be done as in Section 2.2: analogous to [7], in a Lorentz-type gauge-fixing $d * \phi = 0$ (conformally invariant). If there is a conformal gauge-fixing, the quantum theory is conformally invariant, and the Ward identities in Section 5.1 are valid. These computations confirm the naive quantisation rules to first order in c. More generally, higher order corrections to these operations are given by loop calculations in the TOM. Here we sketch the tree-level computation, more details can be found in [60]. It goes exactly analogous to Section 2.7.

We consider all scalars and vectors as fields, while the 2-forms and 3-forms are considered antifields; this is for computational ease.

$$\begin{array}{rcccccccc} \text{Fields:} & X^i & \rho^i & \chi_i & \psi^i & F_i & H_i & A^i & \eta_i, \\ \text{Antifields:} & \eta_{(3)i} & F_{(2)i} & A^i_{(3)} & H_{(3)i} & \rho^i_{(3)} & \psi^i_{(2)} & \chi_{(2)i} & X^i_{(2)}. \end{array} \tag{5.52}$$

There are gauge invariances for the four 1-form fields. We therefore have to introduce four pairs of scalar antighosts.

gauge field	antifield	ghost #	degree	Lagr. mult.	ghost #	degree
ρ^i	P_i	0	0	λ_i	1	0
H_i	π^i	-1	0	L^i	0	0
A^i	γ_i	-1	0	Y_i	0	0
η_i	Z^i	-2	0	σ^i	-1	0

(5.53)

The antifields are replaced by

$$\begin{array}{cccc} F_{(2)i} = *dP_i, & X^i_{(2)} = *dZ^i, & \chi_{(2)i} = *d\gamma_i, & \psi^i_{(2)} = *d\pi^i, \\ H_{(3)i} = 0, & A^i_{(3)} = 0, & \rho^i_{(3)} = 0, & \eta_{(3)i} = 0, \end{array} \tag{5.54}$$

and the extra terms in the action from the antighosts are

$$S_{\text{antighost}} = \int \Big(dL^i * H_i + d\lambda_i * \rho^i + dY_i * A^i + d\sigma^i * \eta_i \Big). \tag{5.55}$$

The kinetic terms in the gauge fixed action are therefore

$$\begin{aligned} S_{\text{kin}} = \int_N \Big(& P_i d * dX^i + F_i d * dZ^i + A^i dH_i + H_i * dL^i + A^i * d(Y_i + P_i) \\ & + \pi^i d * d\chi_i + \psi^i d * d\gamma_i + \eta_i d\rho^i + \rho^i * d\lambda_i + \eta_i * d(\sigma^i + \pi^i) \Big). \end{aligned} \tag{5.56}$$

There are two types of kinetic terms: those for two scalars and those for two vectors and two scalars. The kinetic terms for two scalars is given by the operator $\Delta_0 = *d_2 * d_0 : \Omega^0 \to \Omega^0$. The d_p are the differentials acting on p-forms, $d_p : \Omega^p \to \Omega^{p+1}$. This has finite dimensional kernel, and therefore can be inverted on the fluctuations, is given by

$$\begin{pmatrix} *d_1 & d_0 \\ *d_2 * & 0 \end{pmatrix} : \quad \Omega^1 \oplus \Omega^0 \to \Omega^1 \oplus \Omega^0. \tag{5.57}$$

This matrix operator satisfies

$$\begin{pmatrix} *d_1 & d_0 \\ *d_2 * & 0 \end{pmatrix}^2 = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_0 \end{pmatrix}, \tag{5.58}$$

where $\Delta_0 = *d_2 * d_0$ and $\Delta_1 = d_0 * d_2 * + * d_1 * d_1$ are the Laplacian acting on 0-forms and 1-forms respectively. The propagator is therefore given by

$$\begin{pmatrix} *d_1 & d_0 \\ *d_2 * & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \Delta_1^{-1} * d_1 & \Delta_1^{-1} d_0 \\ \Delta_0^{-1} * d_2 * & 0 \end{pmatrix} = \begin{pmatrix} *d_1 \Delta_1^{-1} & d_0 \Delta_0^{-1} \\ *d_2 * \Delta_1^{-1} & 0 \end{pmatrix}. \tag{5.59}$$

If we assume there are no boundary contributions, the propagators are given by

$$\begin{array}{ccc} \langle X^i P_j \rangle = i\delta^i_j \Delta_0^{-1}, & \langle F_i Z^j \rangle = i\delta^j_i \Delta_0^{-1}, & \\ \langle \chi_i \pi^j \rangle = i\delta^j_i \Delta_0^{-1}, & \langle \psi^i \gamma_j \rangle = i\delta^i_j \Delta_0^{-1}, & \\ \langle H_i A^j \rangle = i\delta^j_i * d_1 \Delta_1^{-1}, & \langle H_i L^j \rangle = i\delta^j_i d_0 \Delta_0^{-1}, & \langle A^i Y_j \rangle = i\delta^i_j d_0 \Delta_0^{-1}, \\ \langle \rho^i \eta_j \rangle = i\delta^i_j * d_1 \Delta_1^{-1}, & \langle \rho^i \lambda_j \rangle = i\delta^i_j d_0 \Delta_0^{-1}, & \langle \eta_i \sigma^j \rangle = i\delta^j_i d_0 \Delta_0^{-1}, \end{array} \tag{5.60}$$

In terms of the gauge fixed superfields

$$\begin{aligned} \boldsymbol{X}^i(x,\theta) &= X^i + \rho^i\zeta + \tfrac{1}{2} *dZ^i\zeta^2, &\qquad \boldsymbol{F}_I(y,\zeta) &= F_i + \eta_i\zeta + \tfrac{1}{2} *dP_i\zeta^2, \\ \boldsymbol{\chi}_i(x,\theta) &= \chi_i + H_i\theta + \tfrac{1}{2} *d\gamma_i\theta^2, &\qquad \boldsymbol{\psi}^i(y,\zeta) &= \psi^i + A^i\zeta + \tfrac{1}{2} *d\pi^i\zeta^2, \end{aligned} \tag{5.61}$$

the relevant propagators for the flat case $N = \mathbb{R}^3$ can be summarised in the form

$$\langle \boldsymbol{X}^i(x,\theta)\boldsymbol{F}_j(y,\zeta)\rangle = i\delta^i_j\mathcal{D}\varphi(x,\theta;y,\zeta), \qquad \langle \boldsymbol{\chi}_i(x,\theta)\boldsymbol{\psi}^i(y,\zeta)\rangle = i\delta^j_i\mathcal{D}\varphi(x,\theta;y,\zeta), \tag{5.62}$$

where $\partial_\mu\partial^\mu\varphi(x,\theta;y,\zeta) = i\delta(x-y)(\theta-\zeta)^3$ and

$$\mathcal{D} = \frac{\partial^2}{\partial\theta_\mu\partial x^\mu} + \frac{\partial^2}{\partial\zeta_\mu\partial y^\mu}. \tag{5.63}$$

One proceeds with imposing suitable boundary conditions and calculating the propagators explicitly. We will not do here, but refer to [60].

The only nontrivial element in the BRST cohomology corresponds to the third cohomology, which is the operator corresponding to the cohomology class of c_{ijk}. This does not deform the bracket on-shell. However, it has an off-shell effect which can also be measured on-shell. It introduces a nonzero third order bracket,

$$\{f,g,h\} = c_{ijk}\frac{\partial f}{\partial\chi_i}\frac{\partial g}{\partial\chi_j}\frac{\partial h}{\partial\chi_k}. \tag{5.64}$$

It is computed from (4.3), which gives

$$\langle\{\chi_i,\chi_j,\chi_k\}(x)\rangle_c = \Big\langle \int_{\partial N} *d\gamma_i(z) \oint_C H_j(y)\,\chi_k(x)\Big\rangle_c \tag{5.65}$$

The relevant part of the true deformation is

$$\frac{1}{6}\int d^3\theta\, \boldsymbol{c}_{ijk}\boldsymbol{\psi}^i\boldsymbol{\psi}^j\boldsymbol{\psi}^k = \boldsymbol{c}_{ijk}\Big(\frac{1}{6}A^iA^jA^k + A^i\psi^j *d\pi^k\Big). \tag{5.66}$$

This gives

$$\langle\{\chi_i,\chi_j,\chi_k\}(x)\rangle_c == \Big\langle \int_N c_{lmn}(\psi^l A^m *d\pi^n)(u) \int_{\partial N} *d\gamma_i(z) \oint_C H_j(y)\,\chi_k(x)\Big\rangle. \tag{5.67}$$

One then substitutes the appropriate contractions and finds

$$\langle\{\chi_i,\chi_j,\chi_k\}(x)\rangle_c = c_{ijk}. \tag{5.68}$$

up to a constant factor which can also be evaluated explicitly (the analogue of the weight w_Γ in Kontsevich' formula). This shows that the deformation is, up to first order in c, indeed of the form (5.50). The c-field deforms the bracket only in the presence of a b-field. This is easily seen from the possible contractions: since ψ has no zero modes (it vanishes on the boundary), it is not integrated out and it needs to be contracted with χ. If b vanishes, the master equation is satisfied trivially; in that case, c does not deform the bracket, but it does deform the L_∞ structure. In a sense the b-field is a representation of the c-field. The b-field is more fundamental for the closed string, it is a gauge field. The c-field is a background field, which is eventually identified with the field strength. However, it is unclear whether "$C = dB$" is to be understood as a classical equation of motion or C is a true field strength.

5.7. Relation between BV and Algebra Structure

In this section we will discuss the relations between the various 2-algebra brackets and the corresponding structure equations, and comment on the relation between the Hochschild cohomology and BV quantisation. The 2-algebra brackets we encountered are the following,

- the Gerstenhaber bracket of closed string operators,
$$[\hat{\phi}_1, \hat{\phi}_2] = \oint \hat{\phi}_1^{(1)} \hat{\phi}_2, \tag{5.69}$$
- the Schouten-Nijenhuis bracket of multivector fields $\phi_1(X, \chi)$ and $\phi_2(X, \chi)$,
$$\{\phi_1, \phi_2\} = \frac{\partial \phi_1}{\partial X^i}\frac{\partial \phi_2}{\partial \chi^i} + (-1)^{|\phi_1|}\frac{\partial \phi_1}{\partial \chi^i}\frac{\partial \phi_2}{\partial X^i}, \tag{5.70}$$
- and the BV bracket of functionals $S_{\phi_1}(\boldsymbol{X}, \boldsymbol{\chi})$ and $S_{\phi_2}(\boldsymbol{X}, \boldsymbol{\chi})$, where $\boldsymbol{X}$ and $\boldsymbol{\chi}$ are superfields,
$$(S_{\phi_1}, S_{\phi_2}) = \frac{\delta S_{\phi_1}}{\delta \boldsymbol{X}^i}\frac{\delta S_{\phi_2}}{\delta \boldsymbol{\chi}^i} - \frac{\delta S_{\phi_1}}{\delta \boldsymbol{\chi}^i}\frac{\delta S_{\phi_2}}{\delta \boldsymbol{X}^i}. \tag{5.71}$$

These relations are as follows. When the closed string algebra is isomorphic to the algebra of multivector fields on a target space M, the Gerstenhaber bracket is equivalent to the Schouten-Nijenhuis bracket. This is the case both in deformation quantisation, where the closed string algebra is the Hochschild cohomology of the algebra of functions, and in the TOM, where the closed string algebra is the quantised boundary algebra of the open membrane. It is seen as follows. Denote the coordinates on the graded tangent space ΠTN of the closed string world-sheet N by x^μ (even) and θ^μ (odd), and the coordinates on ΠT^*M by X^i and χ_i, the latter being the fermionic coordinate representing $\frac{\partial}{\partial X^i}$. Multivector fields are represented by functions $\phi(X, \chi)$. One introduces superfields, polynomials in θ with descendants as coefficients,

$$\boldsymbol{X}^i = X^i + \rho^i\theta + \frac{1}{2}X^{(2)i}\theta^2, \tag{5.72}$$
$$\boldsymbol{\chi}_i = \chi_i + H_i\theta + \frac{1}{2}\chi_i^{(2)}\theta^2, \tag{5.73}$$

where there is a contraction for every θ. The BV action reads

$$S_0 = \int_N \int d^2\theta\, \boldsymbol{\chi}_i D\boldsymbol{X}^i = \int_N H_i dX^i + \chi_i d\rho^i, \tag{5.74}$$

with $D = \theta^\mu \partial_\mu$. H and χ are the canonical conjugates of X and ρ, respectively. The corresponding bracket

$$(\cdot,\cdot) = \frac{\partial}{\partial \boldsymbol{X}^i} \wedge \frac{\partial}{\partial \boldsymbol{\chi}_i} \tag{5.75}$$

is indeed a BV bracket because by construction, the superfields contain the antifields. The derivation then goes as in (4.21) and (5.29),

$$\begin{aligned}[\hat{\phi}_1, \hat{\phi}_2] &= \oint \left(\rho^i \widehat{\frac{\partial\phi_1}{\partial X^i} + H_i \frac{\partial \phi_1}{\partial \chi_i}}\right)\hat{\phi}_2 = \left((-1)^{|\phi_1|}\frac{\partial \phi_1}{\partial X^i}\rho^i + \frac{\partial\phi_1}{\partial \chi_i}H_i\right)\phi_2 \\ &= (-1)^{|\phi_1|}\frac{\partial\phi_1}{\partial X^i}\frac{\partial \phi_2}{\partial \chi^i} + \frac{\partial\phi_1}{\partial \chi^i}\frac{\partial\phi_2}{\partial X^i} = (-1)^{|\phi_1|}\{\phi_1,\phi_2\},\end{aligned}$$

as we wanted to show. In terms of the OPE in closed string theory, the contour integral

$$\oint :\phi_1^{(1)}(X,\chi)(z) :: \phi_2(X,\chi)(w): \tag{5.76}$$

can be written as a commutator $\frac{\partial\phi_1}{\partial X}\frac{\partial\phi_2}{\partial\chi} + (-1)^{|\phi_1|}\frac{\partial\phi_1}{\partial\chi}\frac{\partial\phi_2}{\partial X}$ if and only if the propagator $\langle X\chi\rangle$ has a first order pole $\frac{1}{z-w}$. In the CF model this is seen in (2.72). Note that the correspondence between multivector fields and bulk operators is also natural from the point of view of noncommutative geometry, because topologically, multivector fields correspond with the homology, and in noncommutative geometry this is cyclic cohomology [61]. The latter is almost the same as the Hochschild cohomology, which is the bulk theory.

Then, the BV structure of the two-dimensional (boundary) field theory, parametrised by operations like $\boldsymbol{Q}$, $\triangle$, and $(\cdot,\cdot)$, and the corresponding 2-algebra structure, parametrised by operations like Q, $\triangle$, and $\{\cdot,\cdot\}$, have a direct relation. We collectively denote the map from ΠTN to functions on ΠT^*M, parametrised by $(\boldsymbol{X}^i, \boldsymbol{\chi}_i)$, by $\boldsymbol{\phi}$. Then a function $\mathcal{O}(X,\chi)$ is related directly to the observable $\boldsymbol{\mathcal{O}} = \boldsymbol{\phi}^*\mathcal{O}$. All the operations in bold face are simply the pull-backs by ϕ of the corresponding operations in the 2-algebra. For example, for the bracket this can be stated as follows

$$[\mathcal{O}_1, \mathcal{O}_2] = \int_\Sigma \int d^2\theta (\boldsymbol{\mathcal{O}}_1, \boldsymbol{\mathcal{O}}_2). \tag{5.77}$$

Note that the right-hand side can be viewed as a function of (X,χ). The fermionic integral removes two powers of θ, which have ghost degree 1. Furthermore, the BV bracket always has ghost degree 1. Therefore, the Gerstenhaber bracket on the left-hand side must have ghost degree -1, which is the correct degree for a 2-algebra.

In the case of the 2-algebra deforming the algebra of functions, there are deformation terms in the action as in in (2.80),

$$
\begin{aligned}
S &= S_0 + S_{\boldsymbol{\phi}}, \\
S_{\boldsymbol{\phi}} &= \int_N \int d^2\theta \boldsymbol{\phi}(\boldsymbol{X}, \boldsymbol{\chi}) = \int_N \int d^2\theta \boldsymbol{\phi}^I \boldsymbol{\chi}_I,
\end{aligned}
\tag{5.78}
$$

where I is a multi-index. The mapping was stated in equation (2.82),

$$
(S_{\boldsymbol{\phi}_1}, S_{\boldsymbol{\phi}_2}) = (-1)^{|\phi_1|} S_{\{\phi_1, \phi_2\}}. \tag{5.79}
$$

The classical BV master equation is equivalent to the vanishing of the Schouten-Nijenhuis or Gerstenhaber bracket,

$$
(S_{\boldsymbol{\phi}_1}, S_{\boldsymbol{\phi}_2}) = 0 \Leftrightarrow [\phi_1, \phi_2] = 0 \Leftrightarrow \{\phi_1, \phi_2\} = 0. \tag{5.80}
$$

Hence the classical BV master equation implies the on-shell Maurer-Cartan equation[3], and deformations which preserve the BV structure, as studied for the TOM, automatically satisfy the L_∞ relations.

The correspondence is also valid for the BV operator. Note that the BV operator is deformed along with the bracket, so deformations of a BV structure amount to deformations of a Gerstenhaber structure. The quantum BV master equation does not have an interpretation as a tree-level integrability equation; it appears at higher genus, as in [4]. For the BRST operator the statement is

$$
\boldsymbol{Q}\boldsymbol{\phi}^*(\mathcal{O}) = \boldsymbol{\phi}^*(Q\mathcal{O}), \tag{5.81}
$$

which implies that both BRST operators have ghost degree 1.

On the membrane, similar relations exist between the 3-algebra of the bulk theory and the BV structure in the field theory. There, because the integration involves three Grassmann variables, the degree is shifted by 3, and the BV operator $\triangle$ and Gerstenhaber bracket $\{\cdot,\cdot\}$ in the 3-algebra have degree -2.

The algebra corresponds to the algebra of zeroth descendants: the operators in the boundary field theory are derived from the zeroth decendants, which are identified with the algebra. The operations structure in the algebra – BRST operator, product, and bracket – are determined by the quantum algebra of the full operators. In principle, the full algebraic structure of the quantum theory, including the decendants, can be encoded into the algebraic structure of the algebra. For example, the nontrivial quantum products involving the first descendants is encoded in the bracket, which acts naturally on the zeroth descendants.

[3] The analogy between the two was noted in [62].

Generalising the cohomology computation, one expects that for a d-algebra the superfields are maps $\Pi T N_d \to (\Pi T)^{d-1}(\Pi T^* M)$, that the undeformed topological model in $d+1$ dimensions is equivalent to the topological model in d dimensions on its boundary, and that the BRST cohomology of the former with a bulk background field is the deformation theory of the latter. There are $2d$ generators, in accordance with the doubling of variables in computing the Hochschild cohomology, and with the BV formalism. The physical interpretation of these generalised models is not entirely clear, and probably has to be considered case by case.

Keeping in mind the relation between the field algebra and the function algebra, the correspondence between BV cohomology and Hochschild cohomology seems very natural and generally valid in BV-quantisable theories. To first order, the Hochschild cohomology exactly gives, or is given by, the BV-structure. Moreover, when there is a topological boundary theory, the BV cohomology represents the deformations of this theory.

5.8. Effective Target Space Action

We will comment briefly on the consequence of the deformations we found. The correlation functions determine an effective action in the target space M, which is defined as the generating functional of the correlation functions of the boundary operators. As we saw, the boundary operators are related to functions of X and χ, which can be identified with multivector fields. The physical fields in the effective action correspond to the physical boundary operators in the open membrane theory. These are the operators of ghost degree 2 $\mathcal{B} = \frac{1}{2} B^{ij}(X)\chi_i\chi_j$, which correspond to degree 2 multivector fields. Interpreting the effective action as the generating functional of the correlation functions $F_{a_0 \dots a_n}$ of the open membrane theory gives in general an effective action functional which to first order in c can be written in the form

$$S_{\text{eff}} = \int_{\Pi T^* M} \Big(\frac{1}{2} \mathcal{B} \cdot Q\mathcal{B} + \frac{1}{3} \mathcal{B} \cdot \{\mathcal{B}, \mathcal{B}\} + \frac{1}{4} \mathcal{B} \cdot \{\mathcal{B}, \mathcal{B}, \mathcal{B}\} \Big), \tag{5.82}$$

where we integrate over the zero-modes of X^i and χ_i. Precisely such a form for the action of the closed string field theory was proposed by Zwiebach [4] for the bosonic closed string, which was shown to satisfy the (quantum) master equation. Generalising his proposal for more general closed string field theories, this is of course what it reduces to in the case of the TOM. The integration over χ picks out the top component in terms of the multivector degree, which is nonzero only for $D = 5$. In other dimensions, we cannot consistently truncate to the physical degrees of freedom, and we also have to take into account other non-physical modes. It seems that 5 dimensions is very natural for this action. In this situation the action is an interacting topological field theory which is very reminiscent of Chern-Simons, but with a 2-form gauge field. This is closely related to the way the Chern-Simons action arises in topological open string theory [46], which is exactly the analogue of the derivation we gave here for the open membrane. Notice that this theory is already interacting for $c = 0$, as we still have a cubic term coming from the bracket. The c-field gives a further quartic interaction term.

We can indeed interpret much of the deformation theory in terms of a generalised gauge theory. Let us first go to a representation in terms of differential forms rather than multivector fields. This can be done if we take as a background an invertible b^{ij}, and write the algebra A in terms of $\chi^i = b^{ij}\chi_j$. Indeed functions of X^i and χ^i can be identified with differential forms, if we identify $\chi^i = dX^i$. In this identification, the BRST operator Q, for $c = 0$, is identified with the De Rham differential.

Turning on a boundary operator $B = \frac{1}{2}B^{ij}\chi_i\chi_j$ affects Q. The perturbed BRST operator has the form

$$Q_B = Q + \{B, \cdot\}, \tag{5.83}$$

in terms of the bracket on the algebra of multivector fields. This can be interpreted as a covariant d-operator. Let us now consider what happens if we start from a nonzero c. The unperturbed BRST operator Q has a connection part proportional to c, as well as deformed bilinear and trilinear brackets, as can be seen form (5.50). Now if we turn on a 2-form B, the abstract formula for the deformed BRST operator Q_B is slightly changed due to the presence of the trilinear product,

$$Q_B = Q + \{B, \cdot\} + \frac{1}{2}\{B, B, \cdot\}. \tag{5.84}$$

Moreover, we also find a correction for the bracket proportional to the trilinear bracket,

$$\{\cdot, \cdot\}_B = \{\cdot, \cdot\} + \{B, \cdot, \cdot\}. \tag{5.85}$$

We might interpret this as a covariant bracket. We can repeat much of what we know about gauge theory to this 2-form theory. There is a field strength given by

$$H = QB + \{B, B\} + \{B, B, B\}. \tag{5.86}$$

The equations of motion for the above Chern-Simons like theory require this field strength to vanish. Also, we have gauge invariances of the form

$$\delta_\Lambda B = Q_B\Lambda = Q\Lambda + \{B, \Lambda\} + \{B, B, \Lambda\}. \tag{5.87}$$

The field strength H is gauge covariant in the sense that $\delta_\Lambda H = \{H, \Lambda\}_B$. Note that the gauge transformation of H involves the covariant bracket.

5.9. Concluding Remarks

The precise relation to little string theory, $(2,0)$ CFT, and M5-branes remains to be studied. The effective theory we wrote down seems more natural in 5 dimensions rather than in 6, which might indicate some relation to D4-branes. We may wonder if the relation b to 2-form gauge field and c to a 3-form background is valid on the nose. The deformation for M5-branes should be related to the total field strength $H = dB + C$. For the TOM the 'field strength' h is constrained to vanish, while the field c seems to deform the algebra (or rather b and c combined). A related question is the choice of boundary conditions for the fields. In the last section we chose ψ to vanish at the boundary, leading naturally to multivector fields for the boundary operators. One can also choose Dirichlet boundary conditions for χ instead of ψ. This leads to boundary operators naturally induced by differential forms. Whether any of these choices, or perhaps both, corresponds to a decoupling limit of M5-branes is a question for further research.

Another interesting connection can be found by relating to mathematics. The deformed L_∞ algebra of the TOM that we found, including the trilinear bracket, can be seen to be the structure of a Courant algebroid [63, 64, 65]. This is a certain fibred generalisation of a quasi-Hopf algebra (quantum group), which arose in the study of constrained quantisation. More precisely, the structure we found in the TOM is that of an exact Courant algebroid. In general, exact Courant algebroids are characterised by an element of $H^3(M, \mathbb{R})$. In our language, this corresponds to the deformation c. The construction of this class is rather analogous to the class in $H^2(M, \mathbb{R})$ of a 'local line bundle' (more precisely, an algebroid of the form $TM \oplus \mathbb{R}$). When this second cohomology class is an integral class, this can be extended to a genuine global line bundle. The meaning of integrality for the third cohomology class is still mysterious, and is related to a global object for the Courant algebroid. Suggestions have been made that this should be a gerbe. The relation of the TOM to 2-form gauge theories indeed is very suggestive in that direction.

The algebraic structure of the deformed TOM could also be helpful in finding a 'nonabelian' generalisation of 2-form gauge theories. String theory suggests the existence of these theories in connection with multiple M5-branes. In the case of D-branes, the structure of the noncommutative gauge theory related to deformed open strings and the nonabelian gauge theory related to multiple D-branes is very similar. Analogously, we could expect the structure of multiple M5-branes and deformed M5-branes to be similar in an appropriate sense. There exist more general Courant algebroids which combine the nonabelian structure of Hopf algebras and the fibration structure of the deformed tangent space we found in the TOM. This is also very suggestive for a generalisation.

The 2-form CS theory we found as an effective theory of the TOM in the target space can be used to describe moduli spaces of flat 2-form theories. If the speculation above turns out to be correct, this can be interpreted as the moduli space of flat gerbes.

5.10. Appendix: Double Complexes and Spectral Sequences

In this appendix we shortly discuss double complexes and their cohomology. This was used in the computations in Section 5.5. It is also the mathematics behind the descent equations. For more details see, e.g., [66].

A double complex consists of a set of vector spaces $C^{p,q}$ carrying two degrees, together with two mutually anticommuting coboundary operators d and δ, so $d^2 = \delta^2 = d\delta + \delta d = 0$.[4] The operator δ increases the first degree p by one, and the d increases q by one. We can draw this double complex in a diagram as in (5.88), with the operator δ acting horizontally and d acting vertically.

$$\begin{array}{ccccccccc} C^{0,0} & \stackrel{\delta}{\to} & C^{1,0} & \to & C^{2,0} & \to & C^{3,0} & \to & \cdots \\ {\scriptstyle d}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ C^{0,1} & \to & C^{1,1} & \to & C^{2,1} & \to & C^{3,1} & \to & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ C^{0,2} & \to & C^{1,2} & \to & C^{2,2} & \to & C^{3,2} & \to & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \vdots & & \vdots & & \vdots & & \vdots & & \end{array} \tag{5.88}$$

To any double complex one can canonically connect a complex, where the total degree equals the sum of the two degrees, so that the degree k space of this complex is given by

$$C^k = \bigoplus_{p+q=k} C^{p,q}. \tag{5.89}$$

The total coboundary operator on this complex is given by $D = d + \delta$. The essential property $D^2 = 0$ can easily be checked from the properties of the two coboundary operators. Also, it is clear that it increases the total degree k by one. There is now a very convenient way to calculate the total cohomology $H_D^*(C)$ of this induced complex. The idea is to calculate separately the d and δ cohomology. First one calculates cohomology with respect to d,

$$E_1 = H_d(C). \tag{5.90}$$

[4] One usually considers commuting coboundary operators, introducing extra sign factors in the formulas. It can easily seen however that this is completely equivalent.

This is the first approximation to the total cohomology. The operator δ also induces a coboundary operation on this cohomology, which we also denote by δ. We can now make a better approximation of the total cohomology by taking the cohomology with respect to this coboundary,

$$E_2 = H_\delta(E_1). \tag{5.91}$$

In general however, there may still be a coboundary operator left on the result. This procedure can be repeated, leading to a sequence of complexes E_r with coboundary operators d_r,

$$E_r = H_{d_r}(E_{r-1}), \tag{5.92}$$

with the r'th coboundary operator having degree $(r, 1-r)$. One usually find that E_r becomes stationary after a certain point. This happens for example if the range of one of the bidegrees is finite, so that d_r must vanish for sufficiently large r.

In the spectral sequence we can represent a class in the zeroth term E_0 by a d-closed element α_0. In the first term E_1 we take the cohomology with respect to δ, but in the d-cohomology. This means that α_0 should be δ-closed up to the image of d. A class in E_1 is therefore represented by a pair (α_0, α_1), such that $d\alpha_0 = 0$, and $\delta\alpha_0 = -d\alpha_1$. Now in general, the second term E_2 has a remaining coboundary operator. The coboundary operator acting on the representing element α_0 is given by the class of $\delta\alpha_1$, $d_2[\alpha_0] = [\delta\alpha_1]$. This can be depicted as follows

$$\begin{array}{ccccc} & & & & \vdots \\ & & \alpha_1 & \rightarrow & d_2\alpha_0 \\ & & \downarrow & & \\ \alpha_0 & \rightarrow & d_1\alpha_0 & & \\ \downarrow & & & & \\ 0 & & & & \end{array} \tag{5.93}$$

where d acts vertically and δ acts horizontally. For α to represent a cohomology class in E_2, this requires $d_2\alpha$ to be zero. Remember however that we are still working in the d-cohomology, therefore it only needs to be zero as a class in this cohomology. In other words, it only needs to be zero modulo a d-exact term. This repeats the diagram above until at some point it terminates, when the differential is zero. It gives rise to a sequence of equations,

$$d\alpha_0 = 0, \qquad \delta\alpha_0 = -d\alpha_1, \qquad \delta\alpha_1 = -d\alpha_2, \qquad \delta\alpha_2 = -d\alpha_3, \qquad \cdots. \tag{5.94}$$

These are the same as the familiar descent equations. It is easily checked that the total representative $\alpha = \alpha_0 + \alpha_1 + \cdots$ is closed with respect to total coboundary D.